Spectral Theory for Real Symmetric Matrices

Gil Cohen

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Overview

Eigenvalues

Recall that a nonzero vector ψ is an eigenvector of a matrix $\mathbf{M}$ with eigenvalue λ if

$$\mathbf{M}\psi = \lambda\psi.$$

Equivalently,

- $\lambda\mathcal{I} - \mathbf{M}$ is singular;
- λ is a root of the characteristic polynomial of $\mathbf{M}$, $\det(x\mathcal{I} - \mathbf{M})$.

Quick important corollaries:

- $\mathbf{M}$ has n eigenvalues in $\mathbb{C}$, counted with multiplicities.
- The product of eigenvalues $\prod_i \lambda_i = \det \mathbf{M}$.

The Spectral Theorem

Theorem (The Spectral Theorem)

Let $\mathbf{M}$ *be an* $n \times n$ *real, symmetric matrix. Then there exist* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ *(not necessarily distinct) and* n *mutually orthogonal unit vectors* $\psi_1, \ldots, \psi_n$ *such that* ψ_i *is an eigenvector of* $\mathbf{M}$ *of eigenvalue* λ_i*.*

We will prove the theorem via a sequence of claims.

The Spectral Theorem - Proof

Claim

Let $\mathbf{M}$ *be an* $n \times n$ *real, symmetric matrix. If* ψ_1, ψ_2 *are eigenvectors with different eigenvalues then* $\psi_1^T \psi_2 = 0$.

Proof

Claim

The eigenvalues of a real, symmetric matrix are real.

Proof

The Spectral Theorem - Proof

Definition

Let $\mathbf{M}$ be a real, symmetric $n \times n$ matrix. A subspace $U \subseteq \mathbb{R}^n$ is $\mathbf{M}$-invariant if $\mathbf{M}u \in U$ for all $u \in U$.

Take, for example, U the span of some eigenvectors.

Claim

Let $\mathbf{M}$ *be a real, symmetric* $n \times n$ *matrix. If* U *is* $\mathbf{M}$*-invariant, so is* $U^{\perp}$.

Proof

The Spectral Theorem - Proof

Claim

Let $\mathbf{M}$ *be a real, symmetric* $n \times n$ *matrix. If* $\emptyset \neq U$ *is* $\mathbf{M}$*-invariant then* U *contains a real eigenvector of* $\mathbf{M}$.

Proof

We are now in a position to prove the spectral theorem.

Theorem (The Spectral Theorem; recall)

Let $\mathbf{M}$ *be an* $n \times n$ *real, symmetric matrix. Then there exist* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ *(not necessarily distinct) and* n *mutually orthogonal unit vectors* $\psi_1, \ldots, \psi_n$ *such that* ψ_i *is an eigenvector of* $\mathbf{M}$ *of eigenvalue* λ_i*.*

Proof

Spectral Decomposition

Corollary (Spectral decomposition)

Let $\mathbf{M}$ *be a real, symmetric* $n \times n$ *matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and corresponding orthonormal eigenvectors* $\psi_1, \ldots, \psi_n$. *Then,*

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^T$$

where $\mathbf{U} = (\psi_1, \ldots, \psi_n)$ *and* $\mathbf{\Sigma} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Thinking of $\mathbf{M}$ as an operator, take $\mathbf{x} \in \mathbb{R}^n$ and write $\mathbf{x} = \sum_i c_i \psi_i$ where $\sum_i c_i^2 = \|\mathbf{x}\|_2^2$. We have that

$$\mathbf{Mx} = \sum_i c_i \mathbf{M} \psi_i = \sum_i \lambda_i c_i \psi_i.$$

The spectral decomposition is useful for taking powers

$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^T = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^T$$

$$\mathbf{M}^2 = \mathbf{U}\boldsymbol{\Sigma}^2\mathbf{U}^T = \sum_{i=1}^{n} \lambda_i^2 \psi_i \psi_i^T$$

If $\lambda_i \neq 0$ for all i, then

$$\mathbf{M}^{-1} = \mathbf{U}\boldsymbol{\Sigma}^{-1}\mathbf{U}^T = \sum_{i=1}^{n} \frac{1}{\lambda_i} \psi_i \psi_i^T$$

If $\mathbf{M}$ is singular, we can still define the pseudo-inverse (aka the Moore–Penrose inverse) by

$$\mathbf{M}^{\dagger} = \sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} \psi_i \psi_i^T.$$

Positive (semi)definite matrices

Definition

A real symmetric matrix $\mathbf{M}$ is positive semidefinite (PSD) if all its eigenvalues are non-negative. It is positive definite (PD) if its eigenvalues are strictly positive.

For a PSD $\mathbf{M}$,

$$\mathbf{M} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^T = \sum_{i=1}^{n} \lambda_i \psi_i \psi_i^T$$

$$\sqrt{\mathbf{M}} = \mathbf{U}\sqrt{\mathbf{\Sigma}}\mathbf{U}^T = \sum_{i=1}^{n} \sqrt{\lambda_i} \psi_i \psi_i^T$$

Trace is the sum of eigenvalues

We wish to prove the following corollary.

Corollary

Let $\mathbf{M}$ *be an* $n \times n$ *real, symmetric matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Then,*

$$\mathrm{Tr}(\mathbf{M}) = \sum_{i=1}^{n} \lambda_i .$$

To prove the corollary we start by recalling properties of the determinant.

The determinant

The most basic nontrivial fact about the determinant is that it is multiplicative. That is,

$$\det(\mathbf{MN}) = \det(\mathbf{M})\det(\mathbf{N})$$

From this we can infer

Theorem (The Weinstein-Aronszajn Determinant Identity)

Let $\mathbf{M}$ *be an* $n \times m$ *matrix, and* $\mathbf{N}$ *an* $m \times n$ *matrix. Then,*

$$\det(\mathcal{I} + \mathbf{MN}) = \det(\mathcal{I} + \mathbf{NM}).$$

Hint: consider $\mathbf{A} = \begin{pmatrix} \mathcal{I} & -\mathbf{M} \\ \mathbf{N} & \mathcal{I} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \mathcal{I} & \mathbf{M} \\ \mathbf{0} & \mathcal{I} \end{pmatrix}$.

The determinant

A key observation is that near the identity, the determinant behaves like the trace. Formally,

$$\det(\mathcal{I} + \varepsilon \mathbf{M}) = 1 + \varepsilon \mathrm{Tr}(\mathbf{M}) + O(\varepsilon^2)$$

Proof

Corollary (The cyclic property of the trace function)

Let $\mathbf{M}$ *be an* $n \times m$ *matrix, and* $\mathbf{N}$ *an* $m \times n$ *matrix. Then,*

$$\mathrm{Tr}(\mathbf{MN}) = \mathrm{Tr}(\mathbf{NM}).$$

Proof

Trace is the sum of eigenvalues

We are now in a position to prove

Corollary

Let $\mathbf{M}$ *be an* $n \times n$ *real, symmetric matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$. *Then,*

$$\mathrm{Tr}(\mathbf{M}) = \sum_{i}^{n} \lambda_i.$$

Proof

Cospectral graphs

Graphs G, H with the same sequence of eigenvalues of their respective $\mathbf{M}_G, \mathbf{M}_H$ are called cospectral. Note that isomorphic graphs are cospectral. Indeed, given a permutation π on V denote

$$\mathbf{\Pi}(u, v) = \begin{cases} 1 & \text{if } \pi(u) = v \\ 0 & \text{otherwise} \end{cases}$$

Observe that $\mathbf{\Pi}\mathbf{e}(u) = \mathbf{e}(\pi^{-1}(u))$ and so $\mathbf{M}_{\pi(G)} = \mathbf{\Pi}^T \mathbf{M}_G \mathbf{\Pi}$.

Cospectral graphs

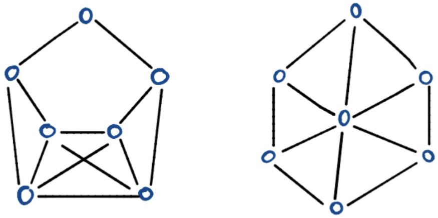
As $\mathbf{M}_{\pi(G)} = \mathbf{\Pi}^T \mathbf{M}_G \mathbf{\Pi}$, if μ an eigenvalue of $\mathbf{M}_G$ with eigenvector $\boldsymbol{\psi}$ then

$$\begin{aligned}
\mathbf{M}_{\pi(G)}(\mathbf{\Pi}^T \boldsymbol{\psi}) &= (\mathbf{\Pi}^T \mathbf{M}_G \mathbf{\Pi})(\mathbf{\Pi}^T \boldsymbol{\psi}) \\
&= \mathbf{\Pi}^T \mathbf{M}_G (\mathbf{\Pi}\mathbf{\Pi}^T) \boldsymbol{\psi} \\
&= \mathbf{\Pi}^T \mathbf{M}_G \boldsymbol{\psi} \\
&= \mu(\mathbf{\Pi}^T \boldsymbol{\psi}).
\end{aligned}$$

Thus, μ is an eigenvalue of $\mathbf{M}_{\pi(G)}$ (note $\mathbf{\Pi}^T \boldsymbol{\psi} \neq 0$).

Cospectral graphs

Cospectral graphs are not necessarily isomorphic.



The adjacency matrices of both graphs have the same characteristic polynomial

$$(x+2)(x+1)^2(x-1)^2(x^2-2x-6)$$

Spectral properties of a graph

We say that a property of a graph is a spectral property if it is determined by its eigenvalues (its spectrum).

Say G is a graph with e edges. As $\mathrm{Tr}(\mathbf{M}_G) = \sum_i \lambda_i$,

$$\sum_i \lambda_i^2 = \mathrm{Tr}(\mathbf{M}_G^2) = 2e.$$

Hence, the number of edges is a spectral property.

Question

What about the number of triangles? 4-cycles? Planarity?

The Fiedler value

The Laplacian of a graph is PSD. We will always sort the eigenvalues of the Laplacian from smallest to largest

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

Lemma

G is connected if and only if $\lambda_2 > 0$.

Proof

λ_2 is called the Fiedler value. Later in the course we will prove a quantitative result known as Cheeger's inequality.

Fiedler's abstract

ALGEBRAIC CONNECTIVITY OF GRAPHS*)

MIROSLAV FIEDLER, Praha

(Received April 14, 1972)

1. INTRODUCTION

Let $G = (V, E)$ be a non-directed finite graph without loops and multiple edges. Having chosen a fixed ordering $w_1, w_2, \ldots, w_n$ of the set V, we can form a square n-rowed matrix $A(G)$ whose off-diagonal entries are $a_{ik} = a_{ki} = -1$ if $(w_i, w_k) \in E$ and $a_{ik} = 0$ otherwise and whose diagonal entries a_{ii} are equal to the valencies of the vertices w_i. This matrix $A(G)$, which is frequently used to enumerate the spanning trees of the graph G, is symmetric, singular (all the row sums are zero) and positive semidefinite ($A(G) = UU^T$ where U is the $(0, 1, -1)$ vertex-edge adjacency matrix of arbitrarily directed graph G). Let $n \geqq 2$ and $0 = \lambda_1 \leqq \lambda_2 = a(G) \leqq \lambda_3 \leqq \ldots \ldots \leqq \lambda_n$ be the eigenvalues of the matrix $A(G)$. From the Perron-Frobenius theorem applied to the matrix $(n-1)I - A(G)$ it follows that $a(G)$ is zero if and only if the graph G is not connected. We shall call the second smallest eigenvalue $a(G)$ of the matrix $A(G)$ algebraic connectivity of the graph G. It is the purpose of this paper to find its relation to the usual vertex and edge connectivities.

We recall that many authors, e.g. A. J. HOFFMAN, M. DOOB, D. K. RAY-CHAUDHURI, J. J. SEIDEL have characterized graphs by means of the spectra of the $(0, 1)$ and $(0, 1, -1)$ adjacency matrices.

Remark. After having finished this paper the author was informed that W. N. ANDERSON, Jr. and T. D. MORLEY had obtained some of these results in the paper Eigenvalues of the Laplacian of a graph, University of Maryland Technical Report TR-71-45, October 6, 1971.

References

[1] *Mc Duffee:* The Theory of Matrices. Springer, Berlin 1933.
[2] *M. Fiedler:* Bounds for eigenvalues of doubly stochastic matrices. Linear Algebra and Its Appl. *5* (1972), 299—310.
[3] *H. Whitney:* Congruent graphs and the connectivity of graphs. Amer. J. Math. *54* (1932,) 150—168.

Author's address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).

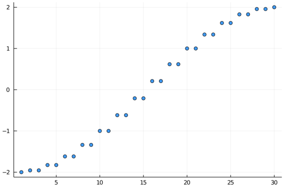
The spectrum of the cycle graph

Lemma

Let G be the cycle graph on $V = [n]$. Let $\omega \in \mathbb{C}$ be an n^{th} root of unity. Then, for every $i \in [n]$,

$$\lambda_i = \omega^i + \omega^{-i}$$

is an eigenvalue of $\mathbf{M}_G$ with eigenvector ψ_i with j^{th} entry $(\psi_i)_j = \omega^{i+j}$.



The Rayleigh quotient

Definition (The Rayleigh quotient)

The Rayleigh quotient of a vector $\mathbf{x}$ with respect to a matrix $\mathbf{M}$ is defined by

$$\frac{\mathbf{x}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Question

What is the Rayleigh quotient of an eigenvector of $\mathbf{M}$*?*

The Rayleigh quotient

Question

What is the largest value that the Rayleigh quotient can attain?

The Rayleigh quotient

Question

Express μ_2 *as a Rayleigh quotient.*

The Courant-Fischer Theorem

Theorem (The Courant-Fischer Theorem)

Let $\mathbf{M}$ *be a symmetric matrix with eigenvalues* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. *Then,*

$$\mu_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim S = k}} \min_{\substack{\mathbf{x} \in S \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$$

$$= \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim T = n-k+1}} \max_{\substack{\mathbf{x} \in T \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

μ_2 as the min of max

Want to show that $\mu_2 = \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim T = n-1}} \max_{\substack{\mathbf{x} \in T \\ \mathbf{x} \neq 0}} \frac{\mathbf{x}^T \mathbf{M} \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$.

Eigenvectors from eigenvalues

Theorem

Let $\mathbf{M}$ *be a symmetric matrix with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *and corresponding real eigenvectors* $\psi_1, \ldots, \psi_n$. *Then,*

$$(\psi_i)_j^2 \cdot \prod_{\substack{k=1 \\ k \neq i}}^{n} (\lambda_i(\mathbf{M}) - \lambda_k(\mathbf{M})) = \prod_{k=1}^{n-1} (\lambda_i(\mathbf{M}) - \lambda_k(\mathbf{M}_j)),$$

where $\mathbf{M}_j$ *is the matrix formed by deleting the* j^{th} *column and row from* $\mathbf{M}$.

See `https://terrytao.wordpress.com/2019/08/13/eigenvectors-from-eigenvalues/` for more information.