Explicit Constructions of Expander Graphs

Following Vadhan, Chapter 4

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Overview

Explicit constructions

Definition

Let G be an undirected graph. We say that G is labelled if every vertex labels its edges by $1, \ldots, \deg(v)$ with no repetitions.

Definition

- A graph G on n vertices is said to be weakly explicit if generating the graph can be done in polynomial-time. That is, if the entire graph can be constructed in time $\text{poly}(n)$.
- G labelled is strongly explicit if accessing any desired neighbor of any vertex can be done in polynomial time. That is, there is an algorithm that given $v \in V$ and $i \in [n]$, returns the i^{th} neighbor of v if $i \leq \deg(v)$, and $\perp$ otherwise. The running time of the algorithm is $\text{poly}(\log n)$.

Squaring

Definition

Let $G = (V, E)$ be an undirected d-regular graph. The square of G, denoted by $G^2 = (V, E')$ is defined as follows. For $(i, j) \in [d]^2$, the $(i, j)^{\text{th}}$ neighbor of a vertex u is the j^{th} neighbor of the i^{th} neighbor of u.

Observe that $\mathbf{M}_{G^2} = \mathbf{M}_G^2$ and $\mathbf{W}_{G^2} = \mathbf{W}_G^2$. Thus, a random step on G^2 is a length-2 random walk on G.

Squaring

Claim

If G is $(1-\omega)$-spectral then G^2 is $(1-\omega^2)$-spectral.

Note that $\gamma(G^2) = 2\gamma(G) - \gamma(G)^2$. Hence, when $\gamma(G)$ is small, γ the spectral gap nearly doubles.

Tensoring

Definition

Let $\mathbf{x}_1 \in \mathbb{R}^{n_1}$, $\mathbf{x}_2 \in \mathbb{R}^{n_2}$. We define the tensor product $\mathbf{x}_1 \otimes \mathbf{x}_2 \in \mathbb{R}^{n_1 n_2}$ of $\mathbf{x}_1, \mathbf{x}_2$ by

$$(\mathbf{x}_1 \otimes \mathbf{x}_2)_{(i_1, i_2)} = (\mathbf{x}_1)_{i_1} (\mathbf{x}_2)_{i_2}.$$

- What is $(\mathbf{x}_1 \otimes \mathbf{x}_2)^T (\mathbf{y}_1 \otimes \mathbf{y}_2)$?
- What is $\|\mathbf{x}_1 \otimes \mathbf{x}_2\|$?

Remark. Not all vectors in $\mathbb{R}^{n_1 n_2}$ are of the form $\mathbf{x} \otimes \mathbf{y}$, though the latter span the space. In particular,

$$\{\mathbf{e}(i) \otimes \mathbf{e}(j) \mid (i, j) \in [n_1] \times [n_2]\}$$

is a basis for $\mathbb{R}^{n_1 n_2}$.

Tensoring

Definition

Let $\mathbf{A}_1$ be an $n_1 \times n_1$ matrix, and $\mathbf{A}_2$ an $n_2 \times n_2$ matrix. The tensor product $\mathbf{A}_1 \otimes \mathbf{A}_2$ is the $(n_1 n_2) \times (n_1 n_2)$ matrix that is defined by

$$(\mathbf{A}_1 \otimes \mathbf{A}_2)_{(i_1,i_2),(j_1,j_2)} = (\mathbf{A}_1)_{i_1,j_1}(\mathbf{A}_2)_{i_2,j_2}.$$

Lemma

$$(\mathbf{A}_1 \otimes \mathbf{A}_2)(\mathbf{x}_1 \otimes \mathbf{x}_2) = (\mathbf{A}_1\mathbf{x}_1) \otimes (\mathbf{A}_2\mathbf{x}_2).$$

Extra space for the proof

Tensoring

Lemma

$$\mathbf{A}_1 \otimes \mathbf{A}_2 = (\mathcal{I}_{n_1} \otimes \mathbf{A}_2)(\mathbf{A}_1 \otimes \mathcal{I}_{n_2}) = (\mathbf{A}_1 \otimes \mathcal{I}_{n_2})(\mathcal{I}_{n_1} \otimes \mathbf{A}_2).$$

Tensoring

- What is $\mathcal{I} \otimes \mathbf{J}$, $\mathbf{J}$ being the normalized all-ones matrix?
- What is $\|\mathbf{A} \otimes \mathbf{B}\|$?

Tensoring

Definition

Let $G_1 = (V_1, E_1)$ be a d_1-regular labelled graph, and $G_2 = (V_2, E_2)$ a d_2-regular graph. Their tensor product is defined by

$$G_1 \otimes G_2 = (V_1 \times V_2, E),$$

as follows. The $(i_1, i_2)^{\text{th}}$ neighbor of (v_1, v_2) is (u_1, u_2) where u_1 is the i_1^{th} neighbor of v_1 in G_1 and u_2 is the i_2^{th} neighbor of v_2 in G_2.

Extra space for a drawing

Tensoring

Observe that

$$\mathbf{M}_{G_1 \otimes G_2} = \mathbf{M}_{G_1} \otimes \mathbf{M}_{G_2}.$$

We further have that

$$\mathbf{W}_{G_1 \otimes G_2} = \mathbf{W}_{G_1} \otimes \mathbf{W}_{G_2}.$$

Thus, a random step on $G_1 \otimes G_2$ consists of a pair of independent random steps on G_1 and G_2.

Tensoring

Lemma

If G_1 is γ_1-spectral and G_2 is γ_2-spectral then, $G_1 \otimes G_2$ is $\min(\gamma_1, \gamma_2)$-spectral.

The vector decomposition method

We will give a second proof for the lemma which will demonstrate the "vector decomposition method".

Given $\mathbf{x} \perp \mathbf{u}_{n_1 n_2}$ decompose it to $\mathbf{x} = \mathbf{x}^{\perp} + \mathbf{x}^{\parallel}$ where $\mathbf{x}^{\parallel}$ is uniform on each cloud and $\mathbf{x}^{\perp}$ is orthogonal to $\mathbf{u}_{n_2}$ on every cloud.

More formally, $\mathbf{x}^{\parallel} = \mathbf{y} \otimes \mathbf{u}_{n_2}$ for some $\mathbf{y} \in \mathbb{R}^{n_1}$ orthogonal to $\mathbf{u}_{n_1}$, and

$$\mathbf{x}^{\perp} = \sum_{i=1}^{n_1} \mathbf{e}(i) \otimes \mathbf{x}_i^{\perp}$$

where each $\mathbf{x}_i^{\perp}$ is orthogonal to $\mathbf{u}_{n_2}$.

The vector decomposition method

We first analyze the operator $\mathbf{W}_1 \otimes \mathbf{W}_2$ on $\mathbf{x}^{\parallel}$.

$$\begin{aligned}(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\parallel} &= (\mathbf{W}_1 \otimes \mathbf{W}_2)(\mathbf{y} \otimes \mathbf{u}_{n_2}) \\ &= (\mathbf{W}_1\mathbf{y}) \otimes (\mathbf{W}_2\mathbf{u}_{n_2}) \\ &= (\mathbf{W}_1\mathbf{y}) \otimes \mathbf{u}_{n_2}.\end{aligned}$$

As $\mathbf{y} \perp \mathbf{u}_{n_2}$,

$$\begin{aligned}\|(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\parallel}\| &= \|(\mathbf{W}_1\mathbf{y}) \otimes \mathbf{u}_{n_2}\| \\ &= \|\mathbf{W}_1\mathbf{y}\|\|\mathbf{u}_{n_2}\| \\ &\le \omega(G_1)\|\mathbf{y}\|\|\mathbf{u}_{n_2}\| \\ &= \omega(G_1)\|\mathbf{y} \otimes \mathbf{u}_{n_2}\| \\ &= \omega(G_1)\|\mathbf{x}^{\parallel}\|.\end{aligned}$$

The vector decomposition method

Next, we analyze the operator $\mathbf{W}_1 \otimes \mathbf{W}_2$ applied to $\mathbf{x}^{\perp}$.

$$\begin{aligned}
(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\perp} &= (\mathbf{W}_1 \otimes \mathcal{I}_{n_2})(\mathcal{I}_{n_1} \otimes \mathbf{W}_2)\mathbf{x}^{\perp} \\
&= (\mathbf{W}_1 \otimes \mathcal{I}_{n_2}) \sum_{i=1}^{n_1} (\mathcal{I}_{n_1} \otimes \mathbf{W}_2)(\mathbf{e}(i) \otimes \mathbf{x}_i^{\perp}) \\
&= (\mathbf{W}_1 \otimes \mathcal{I}_{n_2}) \sum_{i=1}^{n_1} \mathbf{e}(i) \otimes (\mathbf{W}_2 \mathbf{x}_i^{\perp}).
\end{aligned}$$

The vector decomposition method

Now,

$$\begin{aligned}
\Big\| \sum_{i=1}^{n_1} \mathbf{e}(i) \otimes (\mathbf{W}_2 \mathbf{x}_i^{\perp}) \Big\|^2 &= \sum_{i=1}^{n_1} \|\mathbf{e}(i) \otimes (\mathbf{W}_2 \mathbf{x}_i^{\perp})\|^2 \\
&\leq \omega(G_2)^2 \sum_{i=1}^{n_1} \|\mathbf{e}(i) \otimes \mathbf{x}_i^{\perp}\|^2 \\
&= \omega(G_2)^2 \Big\| \sum_{i=1}^{n_1} \mathbf{e}(i) \otimes \mathbf{x}_i^{\perp} \Big\|^2 \\
&= \omega(G_2)^2 \|\mathbf{x}^{\perp}\|^2.
\end{aligned}$$

As $\|\mathbf{W}_1 \otimes \mathcal{I}_{n_2}\| \leq 1$, we conclude $\|(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\perp}\| \leq \omega(G_2)\|\mathbf{x}^{\perp}\|$.

The vector decomposition method

Lastly, we observe that $(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\perp}$ is orthogonal to $(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\parallel}$. Indeed,

$$(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\parallel} = (\mathbf{W}_1\mathbf{y}) \otimes \mathbf{u}_{n_2},$$

and so it is uniform on each cloud, whereas

$$\begin{aligned}(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\perp} &= (\mathbf{W}_1 \otimes \mathcal{I}_{n_2}) \sum_{i=1}^{n_1} \mathbf{e}(i) \otimes (\mathbf{W}_2\mathbf{x}_i^{\perp}) \\ &= \sum_{i=1}^{n_1} (\mathbf{W}_1\mathbf{e}(i)) \otimes (\mathbf{W}_2\mathbf{x}_i^{\perp})\end{aligned}$$

which is orthogonal to $\mathbf{u}_{n_2}$ on each cloud.

The vector decomposition method

Thus,

$$\begin{aligned}\|(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}\|^2 &= \|(\mathbf{W}_1 \otimes \mathbf{W}_2)(\mathbf{x}^{\|} + \mathbf{x}^{\perp})\|^2 \\ &= \|(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\|}\|^2 + \|(\mathbf{W}_1 \otimes \mathbf{W}_2)\mathbf{x}^{\perp}\|^2 \\ &\leq \omega(G_1)^2\|\mathbf{x}^{\|}\|^2 + \omega(G_2)^2\|\mathbf{x}^{\perp}\|^2 \\ &\leq \max(\omega(G_1)^2, \omega(G_2)^2)\|\mathbf{x}\|^2.\end{aligned}$$

Hence, $\omega(G_1 \otimes G_2) \leq \max(\omega(G_1), \omega(G_2))$.

Overview

Recap

To recap,

	Number of vertices	degree	spectral gap
Squaring			
Tensoring			
Zig-Zag			

The Zig-Zag product

Definition

Let $G = (V, E)$ be a d-regular labelled undirected graph. An edge-rotation map is a function $\pi : V \times [d] \to V \times [d]$ such that for every u, i, if $\pi(u, i) = (v, j)$ then the i^{th} neighbor of u is v and the j^{th} neighbor of v is u.

We denote by $\dot{\pi}(u, i)$ the first component of $\pi(u, i)$, namely, the vertex alone.

Observe that π is an involution.

The Zig-Zag product

Definition

- Let G be a d_1 regular undirected graph on n_1 vertices with edge-rotation map π_G.
- Let H be a d_2 regular graph on d_1 vertices with edge-rotation map π_H.

The Zig-Zag product of G, H, denoted by $G ⓩ H$ is the graph whose vertex set is $[n_1] \times [d_1]$. For $a, b \in [d_2]$, the $(a, b)^{\text{th}}$ neighbor of vertex (u, i) is the vertex (v, j) computed as follows:

1. Let $i' = \dot{\pi}_H(i, a)$.
2. Let $(v, j') = \pi_G(u, i')$.
3. Let $j = \dot{\pi}_H(j', b)$.

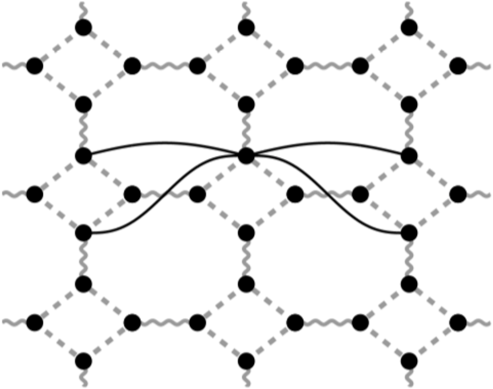

Figure: The Zig-Zag product of the grid $\mathbf{Z}^2$ with the 4-cycle. Figure shamelessly taken from the Hoory-Linial-Wigderson excellent survey entitled "Expander Graphs and Their Applications".

The Zig-Zag product - analysis

Theorem

If G is a γ_G-spectral expander and H a γ_H-spectral expander then $G \textcircled{z} H$ is a $\gamma_H^2 \gamma_G$-spectral expander.

Let P be the permutation (involution even) matrix with

$$\mathbf{P}_{(u,i),(v,j)} = \begin{cases} 1 & \pi_G(u,i) = (v,j); \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{\mathbf{M}}_H = \mathcal{I}_{n_1} \otimes \mathbf{M}_H$ and denote by $\mathbf{M}$ the adjacency matrix of $G \textcircled{z} H$.

The Zig-Zag product - the never proven claim

Claim (The never proven claim)

$$\mathbf{M} = \tilde{\mathbf{M}}_H \mathbf{P} \tilde{\mathbf{M}}_H.$$

The above claim is very intuitive and annoying to write down formally. But, we will do so to practice our tensoring skills. To prove the claim, first recall that, generally, if **P** is a permutation matrix representing a permutation π, namely,

$$\mathbf{P}_{a,b} = \begin{cases} 1 & \pi(a) = b; \\ 0 & \text{otherwise,} \end{cases}$$

then $\mathbf{Pe}(a) = \mathbf{e}(\pi^{-1}(a))$. When **P** is an involution, we get $\mathbf{Pe}(a) = \mathbf{e}(\pi(a))$.

The Zig-Zag product - the never proven claim

Lets spell out what is it we want to prove. We wish to show that $(\tilde{\mathbf{M}}_H \mathbf{P} \tilde{\mathbf{M}}_H)_{(u,i),(v,j)} = 1$ if and only if there exist $a, b \in [d_2]$ such that if we denote $i' = \dot{\pi}_H(i, a)$ and compute $(v, j') = \pi_G(u, i')$ then $j = \dot{\pi}_H(j', b)$. Now,

$$\begin{aligned} \tilde{\mathbf{M}}_H \mathbf{e}(u, i) &= (\mathcal{I}_{n_1} \otimes \mathbf{M}_H)(\mathbf{e}(u) \otimes \mathbf{e}(i)) \\ &= \mathbf{e}(u) \otimes (\mathbf{M}_H \mathbf{e}(i)) \\ &= \mathbf{e}(u) \otimes \sum_{a'=1}^{d_2} \mathbf{e}(\dot{\pi}_H(i, a')) \\ &= \mathbf{e}(u, i') + \mathbf{e}(u) \otimes \mathbf{r}, \end{aligned}$$

where $\mathbf{r}(i') = 0$.

The Zig-Zag product - the never proven claim

Now,

$$\mathbf{Pe}(u, i') = \mathbf{e}(\pi_G(u, i')) = \mathbf{e}(v, j'),$$

whereas $\mathbf{P}(\mathbf{e}(u) \otimes \mathbf{r})$ iz zero on all entries $(v, \cdot)$.

Considering the third step,

$$\begin{aligned}\tilde{\mathbf{M}}_H \mathbf{e}(v, j') &= \mathbf{e}(v) \otimes \sum_{b'=1}^{d_2} \mathbf{e}(\dot{\pi}_H(j', b')) \\ &= \mathbf{e}(v, \dot{\pi}_H(j', b)) + \mathbf{s} \\ &= \mathbf{e}(v, j) + \mathbf{s}\end{aligned}$$

where $\mathbf{s}(v, j) = 0$.

The Zig-Zag product - the never proven claim

Moreover, $\tilde{\mathbf{M}}_H \mathbf{P}(\mathbf{e}(u) \otimes \mathbf{r})$ is also zero on $(v, \cdot)$. Thus, $(\tilde{\mathbf{M}}_H \mathbf{P} \tilde{\mathbf{M}}_H)_{(u,i),(v,j)} = 1$ when a, b as above exist. The proof then follows by a counting argument: the degree of $\tilde{\mathbf{M}}_H \mathbf{P} \tilde{\mathbf{M}}_H$ is d_2^2 - the same number of choices for a, b.

This proves the never proven claim (who now needs a new name).

The Zig-Zag product - analysis

Going back to the analysis of the Zig-Zag product, we have that $\mathbf{M} = \tilde{\mathbf{M}}_H \mathbf{P} \tilde{\mathbf{M}}_H$, and hence, by regularity,

$$\mathbf{W} = \tilde{\mathbf{W}}_H \mathbf{P} \tilde{\mathbf{W}}_H,$$

where $\mathbf{W}$ is the random walk matrix of $G \textcircled{z} H$.

Recall that $\mathbf{W}_H = \gamma_H \mathbf{J} + \omega_H \mathbf{E}_H$ where $\|\mathbf{E}_H\| \leq 1$. Thus,

$$\begin{aligned} \tilde{\mathbf{W}}_H &= \mathcal{I}_{n_1} \otimes (\gamma_H \mathbf{J} + \omega_H \mathbf{E}_H) \\ &= \gamma_H \tilde{\mathbf{J}} + \omega_H \tilde{\mathbf{E}}_H, \end{aligned}$$

where $\tilde{\mathbf{J}} = \mathcal{I}_{n_1} \otimes \mathbf{J}$ and $\tilde{\mathbf{E}}_H = \mathcal{I}_{n_1} \otimes \mathbf{E}_H$.

The Zig-Zag product - analysis

Hence,

$$\tilde{\mathbf{W}} = \gamma_H^2 \tilde{\mathbf{J}}\mathbf{P}\tilde{\mathbf{J}} + \widehat{\mathbf{E}},$$

where

$$\widehat{\mathbf{E}} = \gamma_H \omega_H \left(\tilde{\mathbf{J}}\mathbf{P}\tilde{\mathbf{E}}_H + \tilde{\mathbf{E}}_H \mathbf{P}\tilde{\mathbf{J}} \right) + \omega_H^2 \tilde{\mathbf{E}}_H \mathbf{P} \tilde{\mathbf{E}}_H.$$

Note that $\|\widehat{\mathbf{E}}\| \leq 2\gamma_H \omega_H + \omega_H^2 = 1 - \gamma_H^2$.

The Zig-Zag product - analysis

The key observation is that

Claim

$$\tilde{\mathbf{J}}\mathbf{P}\tilde{\mathbf{J}} = \mathbf{W}_G \otimes \mathbf{J}.$$

The Zig-Zag product - analysis

To recap,

$$\tilde{\mathbf{W}} = \gamma_H^2(\mathbf{W}_G \otimes \mathbf{J}) + \widehat{\mathbf{E}},$$

where $\|\widehat{\mathbf{E}}\| \leq 1 - \gamma_H^2$. For every $\mathbf{x} \perp \mathbf{1}$,

$$\begin{aligned} \|\tilde{\mathbf{W}}\mathbf{x}\| &\leq \gamma_H^2 \|(\mathbf{W}_G \otimes \mathbf{J})\mathbf{x}\| + \|\widehat{\mathbf{E}}\mathbf{x}\| \\ &\leq \gamma_H^2(1 - \gamma_G) + 1 - \gamma_H^2 \\ &= 1 - \gamma_H^2 \gamma_G. \end{aligned}$$

Extra space for the proof

Recap

To recap,

	Number of vertices	degree	spectral gap
Squaring			
Tensoring			
Zig-Zag			

Weakly explicit construction

Let H be a d-regular $\frac{7}{8}$-spectral expander on d^4 vertices. Specifically, one can take the expander you constructed in the problem set, based on Cayley graphs (alternatively, brute force search).

We iteratively construct graphs $G_1, G_2, \ldots$ where

$$G_1 = H^2$$
$$G_{t+1} = G_t^2 \textcircled{z} H.$$

Proposition

For every t, G_t is a d^2-regular $\frac{1}{2}$-spectral expander on d^{4t} vertices.

Extra space for the proof

Fully explicit, yet scarce, construction

Iteratively construct graphs $G_1, G_2, \ldots$ where

$$\begin{aligned} G_1 &= H^2 \\ G_{t+1} &= (G_t \otimes G_t)^2 \text{Ⓩ} H. \end{aligned}$$

Though now we take H to be on d^8 vertices.

Extra space for the proof

Fully explicit construction

The downside of the above suggestion is that the family is rather scarce. To overcome this, consider the variant in which

$$G_1 = H^2$$
$$G_{t+1} = (G_{\lceil t/2 \rceil} \otimes G_{\lfloor t/2 \rfloor})^2 \, Ⓩ \, H.$$

Here the number of vertices increases only exponentially with t (as opposed to double-exponentially) though the recursive relation regarding time improves from

$$T(t) = 2T(t-1) + (\log n)^{O(1)}$$

to about

$$T(t) = 2T(t/2) + (\log n)^{O(1)}.$$

How close to Ramanujan do we get?

How close to Ramanujan do we get? The degree is $D = d^2$ and H is a d-regular graph. By taking H Ramanujan, we get

$$\omega = O\left(\frac{1}{\sqrt{d}}\right) = O\left(\frac{1}{D^{1/4}}\right).$$

In the unit after the next one, we will improve this to $O(1/D^{1/2-o(1)})$.