Cauchy's interlacing theorem

Following Godsil-Royle, Chapters 8.13, 9.1, 13.6

Gil Cohen

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Overview

Cofactor, adjugate and the determinant

Let $\mathbf{A}$ be an $n \times n$ real matrix. We denote by $\mathbf{A}_{(i,j)}$ the submatrix of A obtained by deleting the i^{th} row and j^{th} column. The (i,j)-minor of A is defined by $\mathbf{M}_{i,j} = \det \mathbf{A}_{(i,j)}$.

The cofactor matrix of $\mathbf{A}$ is the $n \times n$ matrix with (i,j)-entry

$$\mathbf{C}_{i,j} = (-1)^{i+j}\mathbf{M}_{i,j}.$$

The adjugate of $\mathbf{A}$ is $\mathrm{adj}(\mathbf{A}) = \mathbf{C}^T$.

Proposition

Let $\mathbf{A}$ *be a square matrix. Then,*

$$\mathbf{A}\mathrm{adj}(\mathbf{A}) = \mathrm{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathcal{I}$$

Corollary

Let $\mathbf{A}$ *be an invertible matrix. Then,*

$$\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{A}^{-1}$$

The resolvent

Definition

Let $\mathbf{A}$ be a real symmetric $n \times n$ matrix. The resolvent of $\mathbf{A}$ is defined by $(xI - \mathbf{A})^{-1}$.

Claim

Let $\mathbf{A}$ be a real symmetric $n \times n$ matrix. If $\mu_1, \ldots, \mu_n$ are the eigenvalues of a $\mathbf{A}$ with corresponding eigenvectors $\psi_1, \ldots, \psi_n$ then

$$(xI - \mathbf{A})^{-1} = \sum_{i=1}^{n} \frac{1}{x - \mu_i} \psi_i \psi_i^T$$

The resolvent

Claim

Let $\mathbf{M}$ *be a real symmetric* $n \times n$ *matrix, and* $\mathbf{N}$ *a matrix obtained by deleting the* i^{th} *row and column of* $\mathbf{M}$. *Then,*

$$\frac{\phi_{\mathbf{N}}(x)}{\phi_{\mathbf{M}}(x)} = e(i)^T (x\mathcal{I} - \mathbf{M})^{-1} e(i).$$

Extra space for the proof

Combinatorial meaning of the derivative of ϕ_G

For an undirected graph $G = (V, E)$, we denote by $G - v$ the graph obtained by deleting v from G. Let ϕ_G denote the characteristic polynomial of $\mathbf{M}_G$.

You will be asked to prove in the problem set that

Lemma

For every an undirected graph $G = (V, E)$,

$$\phi'_G(x) = \sum_{v \in V} \phi_{G-v}(x).$$

Overview

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Claim

Let $\mathbf{M}$ *be a real symmetric* $n \times n$ *matrix, and* $\boldsymbol{b} \in \mathbb{R}^n$*. Define*

$$\psi(x) = \boldsymbol{b}^T (x\boldsymbol{\mathcal{I}} - \mathbf{M})^{-1} \boldsymbol{b}.$$

Then,

1. *All zeros and poles of* ψ *are simple.*
2. ψ' *is negative whenever it is defined.*
3. *If* $p_1 < p_2$ *are two consecutive poles of* ψ*, the closed interval* $[p_1, p_2]$ *contains exactly one zero of* ψ*.*

Example

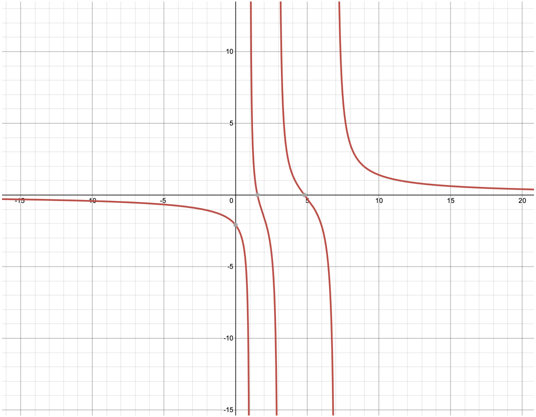

Figure: Plot of $\frac{1}{x-1} + \frac{2}{x-3} + \frac{3}{x-7}$.

Cauchy's interlacing theorem

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *real symmetric matrix with eigenvalues* $\alpha_1 \geq \cdots \geq \alpha_n$. *Let* $\mathbf{B}$ *a principal submatrix of* $\mathbf{A}$ *of dimension* $n-1$ *with eigenvalues* $\beta_1 \geq \cdots \geq \beta_{n-1}$. *Then,*

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n.$$

Extra space for the proof

Applications to the Laplacian

Proposition

Let G be an undirected graph and H obtained by adding an edge to G. Then, for every $1 \leq i < n$,

$$\lambda_i(G) \leq \lambda_i(H) \leq \lambda_{i+1}(G).$$

Extra space for the proof

Applications to the Laplacian

Proposition

Let G be an undirected graph and H obtained by adding an edge to G. Then,

$$\lambda_2(G) \leq \lambda_2(H) \leq \lambda_2(G) + 2.$$

This will probably be left for you to prove on the problem set, also investigating when the right inequality is tight.

A final remark on eigenvectors from eigenvalues

These are "bonus" slides for those who took complex analysis. Recall

$$(zI - \mathbf{A})^{-1} = \sum_{i=1}^{n} \frac{1}{z - \mu_i} \psi_i \psi_i^T$$

Using Cauchy residue formula, if μ_k is isolated and γ a contour that goes only around μ_k we get

$$\oint_{\gamma} (z\mathcal{I} - \mathbf{A})^{-1} dz = 2\pi i \psi_k \psi_k^T .$$

So perhaps it is not so surprising that knowing the spectrum of **A** allows us, in principle, to obtain information about the eigenvectors.

A final remark on eigenvectors from eigenvalues

By a slight tweak,

$$\oint_\gamma (z\mathcal{I} - \mathbf{A})^{-1} z dz = 2\pi i \mu_k \psi_k \psi_k^T.$$

Hence,

$$\frac{1}{2\pi i} \oint_\gamma \mathrm{Tr}((z\mathcal{I} - \mathbf{A})^{-1} z) dz = \mu_k.$$