Rate amplification and query-efficient distance amplification for linear LCC and LDC

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Benny Chor

To the memory of Benny Chor

Overview

Error correcting codes

Definition (Error correcting code)

A set $C \subseteq \Sigma^n$ s.t.

$$\text{dist}(x, y) = |\{i \in [n] : x_i \neq y_i\}| \geq \delta n$$

$\forall x \neq y \in C$ is a code with distance δ.

- We focus on linear codes in which $\Sigma = \mathbb{F}$ is a finite field and C an $\mathbb{F}$-vector space. In such case, the rate of C is given by

$$\rho = \frac{1}{n} \dim C.$$

- A linear code can be defined as a linear map $c : \mathbb{F}^k \to \mathbb{F}^n$ s.t. $C = c(\mathbb{F}^k)$.

Error correcting codes

- A family of codes $\{c : \mathbb{F}^k \to \mathbb{F}^{n(k)}\}_k$ is asymptotically good if both $\delta, \rho = \Omega(1)$.
- Good codes exist.
- A code can be used for communication over imperfect channels; Instead of sending a message $m \in \mathbb{F}^k$ one sends $c(m)$. If the fraction of errors $< \frac{\delta}{2}$, the receiver can infer m.

What if we are only interested in m_i for some $i \in [k]$?

Locally decodable codes

Definition (Locally decodable codes; Katz-Trevisan 2000)

A code $c : \mathbb{F}^k \to \mathbb{F}^n$ is a q-query locally decodable code (LDC) if there is a randomized algorithm

$$D : \mathbb{F}^n \times [k] \to \mathbb{F}$$

(called a decoder) s.t. $\forall i \in [k]$, $m \in \mathbb{F}^k, r \in \mathbb{F}^n$,

$$\mathrm{dist}(r, c(m)) \leq \delta n \quad \Longrightarrow \quad \Pr[D(r, i) = m_i] \geq \frac{2}{3}.$$

Further, the number of queries made by D to r is bounded by q.

Locally correctable codes

Definition (Locally correctable codes)

A code $C \subseteq \mathbb{F}^n$ is a q-query locally correctable code (LCC) if there is a randomized algorithm

$$D : \mathbb{F}^n \times [n] \to \mathbb{F}$$

(called a corrector) s.t. $\forall i \in [n]$, $c \in C$, $r \in \mathbb{F}^n$,

$$\text{dist}(c, r) \leq \delta n \quad \Longrightarrow \quad \Pr[D(r, i) = c_i] \geq \frac{2}{3}.$$

Further, the number of queries made by D to r is at most q.

We will focus on non-adaptive LC in which the choice of which indices the query is done prior to the actual querying.

What is known?

- Much work has been done on the constant query regime.
- More recently, good LC were considered, where we wish to minimize the query complexity.
- Katz-Trevisan proved $q = \Omega(\log n)$ for LDC.
- Linear LCC $\implies$ linear LDC.
- LC need structure - random won't do.

What is the lowest query complexity $q = q(n)$ of good codes?

Explicit constructions

- Reed-Muller codes give $q = n^{\varepsilon}$ for any desired constant ε.
- For a while, no (nontrivial) LC were known with $\rho > \frac{1}{2}$.
- The first construction to break the "rate $\frac{1}{2}$ barrier" is via multiplicity codes (Kopparty-Saraf-Yekhanin 2010).
- Alternative constructions were obtained using expanders (Hemenway-Rafail-Ostrovsky-Wootters 2013) and using lifted RM (Guo-Kopparty-Saraf 2013). All require $q = n^{\varepsilon}$.

Explicit constructions

- The state of the art construction of LC (Kopparty-Meir-Ron-Zewi-Saraf 2016) achieves

$$q = 2^{\widetilde{O}(\sqrt{\log n})} = n^{o(1)}.$$

- Spoiler: In this work we don't improve upon this great result (nor show it is tight).

This work's theme

bad LC $\implies$ good LC.

Overview

Reed-Muller Codes

As an example, we describe an LCC that works assuming no errors, having query complexity $\sqrt{n}$.

Let $\mathbb{F}_q$ be finite field. Define

$$C = \{f(\mathbb{F}_q^2) \mid f \in \mathbb{F}_q^{<q-1}[x, y]\}.$$

We have $n = q^2$ and

$$k = \frac{q(q-1)}{2} \approx \frac{n}{2}.$$

Hence, $\rho = \frac{1}{2} - o(1)$.

Reed-Muller Codes

The corrector. Given a desired index $p = (a, b) \in \mathbb{F}_q^2$, sample a random line through p,

$$L = \{(At + a, Bt + b) \mid t \in \mathbb{F}_q\},$$

and query on $L \setminus p$.

Analysis. Set $g(t) = f(At + a, Bt + b)$. As $\deg g < q - 1$, the $q - 1$ points we query determine g. Thus, we can compute

$$g(0) = f(a, b) = f(p).$$

Reed-Muller Codes

- As Nati pointed out, if we assume no errors then we have a trivial corrector with $q = 1$ queries.
- The above LCC however has the advantage of being smooth: The marginal distribution of every query is uniform over $\mathbb{F}_q^2 \setminus p$.
- Observe that a smooth LCC has distance $\Omega(\frac{1}{q}) = \Omega(\frac{1}{\sqrt{n}})$.
- In any case, to get a good code, one can restrict to degree αq for a constant $\alpha < 1$.

Multiplicity codes

A first example of multiplicity codes is given by

$$(\mathbb{F}_q^3)^n \supseteq C = \{\left(f(\mathbb{F}_q^2), f_x(\mathbb{F}_q^2), f_y(\mathbb{F}_q^2)\right) \mid f \in \mathbb{F}_q^{<d}[x,y]\}$$

with $d = 2q-2$. The rate is then $\rho \approx \frac{2}{3}$.

Corrector & Analysis.

- Pass three random lines through p. Each gives a linear relation between $f(p), f_x(p), f_y(p)$.
- W.h.p the relations are linearly independent.
- Solve a system of linear equations to get $(f(p), f_x(p), f_y(p))$.

One can consider higher dimension and higher order derivatives.

KMRZS and the AEL distance amplification procedure

The KMRZS construction consists of two steps:

1. Use a multiplicity code with $\delta = \frac{1}{\text{poly}\log n}$ and $q = 2^{\widetilde{O}(\sqrt{\log n})}$.
2. Amplify the distance to a constant.

For (2), the AEL distance amplification procedure is invoked (Alon-Edmonds-Luby 1995) that converts a q-query LC with distance δ to a

$$q_{\text{new}} = q \cdot \text{poly}\frac{1}{\delta}$$

query LC with constant distance.

Overview

Query-efficient distance amplification

$$\text{bad LC} \implies \text{good LC}.$$

Theorem (Query-efficient distance amplification)

One can efficiently transform a q-query LDC with distance δ to a constant distance LDC with query complexity

$$q_{\text{new}} = q \cdot \log(n) \cdot q\left(\frac{1}{\delta}\right),$$

where $q(b)$ is the query complexity of a good LDC on message length b.

Query-efficient distance amplification

Corollary

For any constant $\alpha < 1$ a q-query LDC with distance $\delta = \frac{1}{n^\alpha}$ can be transformed to a constant distance LDC with query complexity

$$q_{\text{new}} = q^{O(\log \log n)}.$$

As another corollary, one can amplify

$$\delta = 2^{-(\log n)^\alpha} \quad \Rightarrow \quad q_{\text{new}} = q^{O(\log \log \log n)}$$

Rate amplification

Our second contribution is a rate amplification procedure.

Theorem (Rate amplification for LCC)

Let $\rho_{\text{desired}} < 1$ constant. Let C be a constant distance q-query LCC with rate ρ. One can transform C to a good LCC with $\rho \geq \rho_{\text{desired}}$ and query complexity

$$q_{\text{new}} = (q \cdot \log n)^{\text{poly}\left(\frac{1}{\rho}\right)}.$$

We will discuss only this result in the remaining of this talk.

Overview

A dual view on Reed-Muller

Claim

If $g \in \mathbb{F}_q[t]$ *is non constant and* $\deg g < q - 1$ *then*

$$\sum_{t \in \mathbb{F}_q} g(t) = 0.$$

Recall that

$$L_{a,b,A,B} = \{(At + a, Bt + b) \mid t \in \mathbb{F}_q\}.$$

If we assume $A \neq 0$ it suffices to consider

$$L_{a,b} = \{(t, at + b) \mid t \in \mathbb{F}_q\}.$$

A dual view on Reed-Muller

Let

$$\mathbf{1}_{a,b}(p) = \begin{cases} 1, & p \in L_{a,b} \\ 0, & \text{otherwise.} \end{cases}$$

We can write

$$\mathbf{1}_{a,b}(x,y) = 1 - (y - (ax+b))^{q-1}.$$

We have that

$$(y - (ax+b))^{q-1} = \sum_{k=0}^{q-1} \sum_{j=0}^{k} (-1)^k \binom{q-1}{k} \binom{k}{j} a^j b^{k-j} x^j y^{q-1-k}.$$

A dual view on Reed-Muller

$$(y-(ax+b))^{q-1} = \sum_{k=0}^{q-1}\sum_{j=0}^{k} c_{k,j}\cdot a^j b^{k-j}\cdot x^j y^{q-1-k}.$$

Observe that having $\{x^j y^i \mid i+j \leq q-1\}$ in $C^\perp$ would suffice for the RM corrector. Now,

$$\dim C^\perp = \frac{q(q+1)}{2}$$

implying

$$\dim C = q^2 - \dim C^\perp = \frac{q(q-1)}{2}.$$

Thence, $C^\perp$ is the dual of RM.

Overview

Structure theorem for LCC

In the dual view on RM, we have

1. The point set $P = \mathbb{F}_q^2$.
2. With every $p \in P$ we associated a distribution D_p over $\mathbb{F}_q^P$ s.t.

$$\Pr_{f \sim D_p}[f(p) = 0] = 0.$$

   Moreover, $|f| = |\{p' \mid f(p') \neq 0\}| \leq q$.
3. $\forall p \neq p' \in P$

$$\Pr_{f \sim D_p}[f(p') \neq 0] \leq \frac{q-1}{q^2-1} = \frac{1}{q+1}.$$

4. $\dim \mathrm{Span} L \leq \frac{q(q+1)}{2}$ where $L = \bigcup_{p \in P} \mathrm{supp}(D_p)$.

Structure theorem for LCC

Definition (Dual LCC)

A (q, τ, ρ)-Dual LCC is a family of distributions $\mathcal{D} = \{D_p\}_{p \in P}$ s.t.

1. $\forall p \in P$
$$\Pr_{f \sim D_p}[f(p) = 0] = 0.$$
2. $\forall f \in L \quad |f| \leq q$, where $L = \bigcup_{p \in P} \mathrm{supp}(D_p)$
3. $\forall p \neq p' \in P$,
$$\Pr_{f \sim D_p}[f(p') \neq 0] \leq \tau.$$
4. $\dim \mathrm{Span} L \leq (1 - \rho)|P|$.

Structure theorem for LCC

Definition (Dual LCC)

A (q, τ, ρ)-Dual LCC is a family of distributions $\mathcal{D} = \{D_p\}_{p \in P}$ s.t.

1. $\Pr_{f \sim D_p}[f(p) = 0] = 0$.
2. $|f| \leq q$.
3. $\Pr_{f \sim D_p}[f(p') \neq 0] \leq \tau$.
4. $\dim \mathrm{Span} L \leq (1 - \rho)|P|$.

We generalize the definition and allow L to contain further elements other than $L_0 = \cup_p \mathrm{supp} D_p$.

Theorem (Informal)

C is a q-query linear LCC with distance d $\iff$ $C^\perp$ is a $(q+1, \tau = \frac{q}{d}, \rho = \frac{1}{n} \dim C)$ dual LCC.

Overview

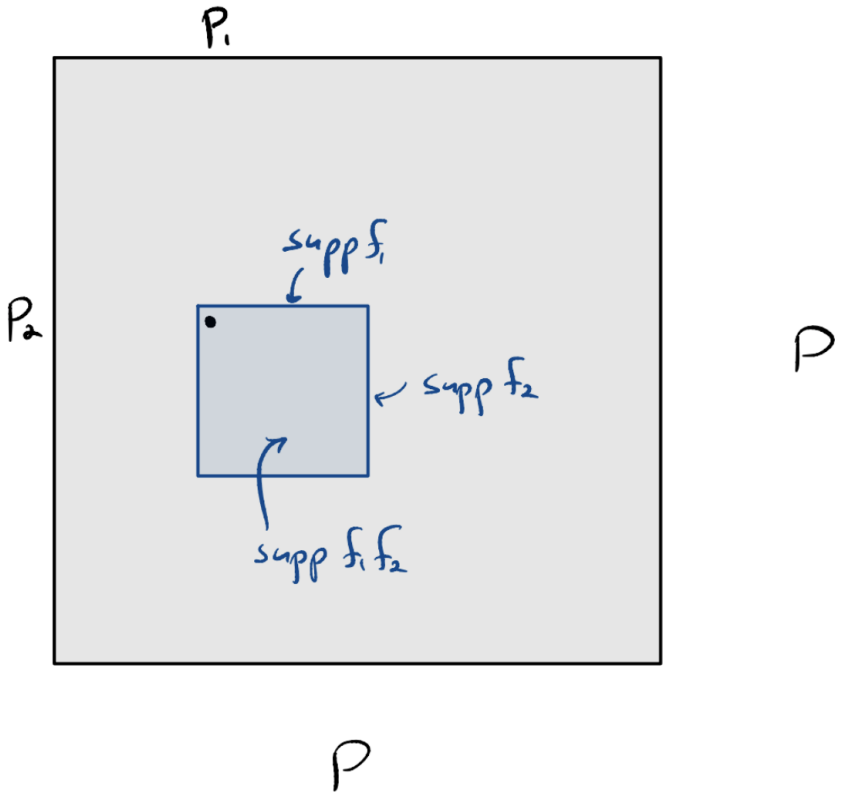
Rate amplification for LCC

The natural way of amplifying the rate of an LCC is to tensor the corresponding dual LCC with itself.

Given a (q, τ, ρ)-Dual LCC $(L, \mathcal{D})$ with $L \subseteq \mathbb{F}^P$ we define $(L', \mathcal{D}')$ as follows:

- The point set is P^2.
- $\mathcal{D}' = \{D'_{(p_1,p_2)} \mid (p_1, p_2) \in P^2\}$ where to sample $f \sim D'_{p_1,p_2}$ we do the following:
  1. Sample $f_1 \sim D_{p_1}$, $f_2 \sim D_{p_2}$ independently.
  2. Output $f_1 f_2$ that is defined by $f_1 f_2(p'_1, p'_2) = f_1(p'_1) f_2(p'_2)$.
- $L' \subseteq \mathbb{F}^{P^2}$ is the smallest subspace containing all functions supported by $\mathcal{D}'$.

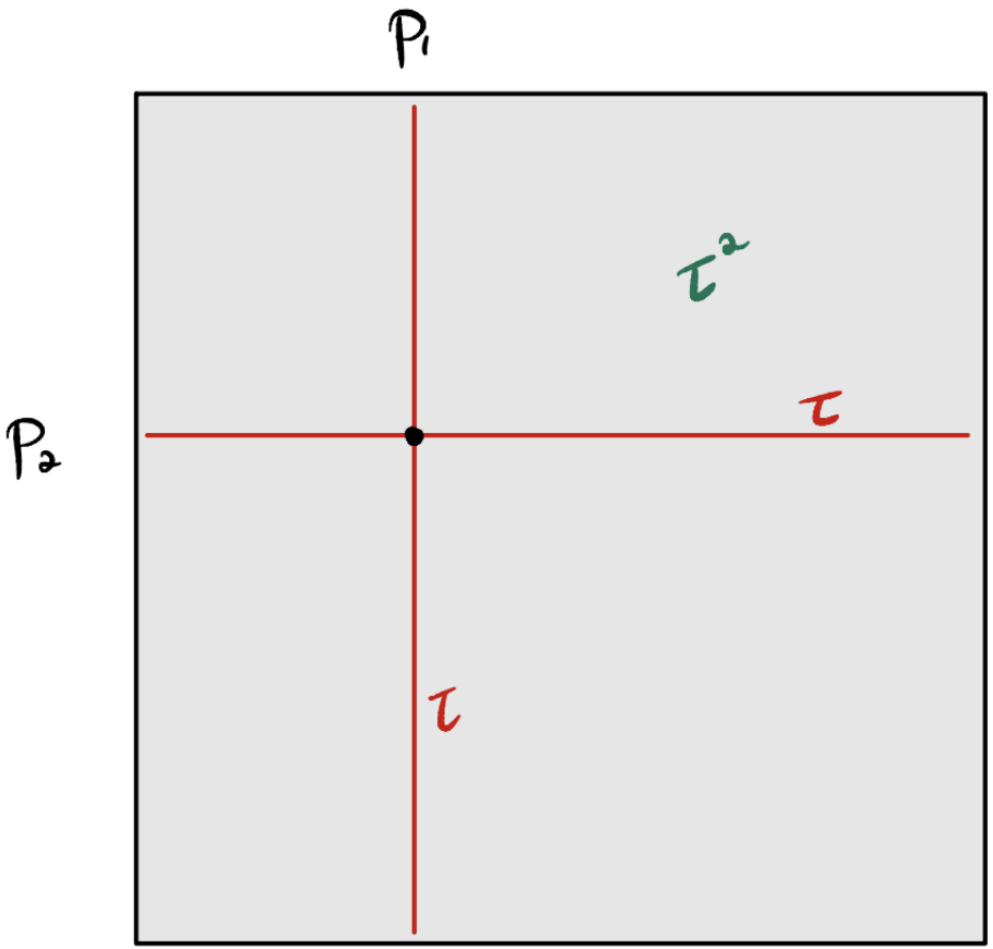
Rate amplification for LCC



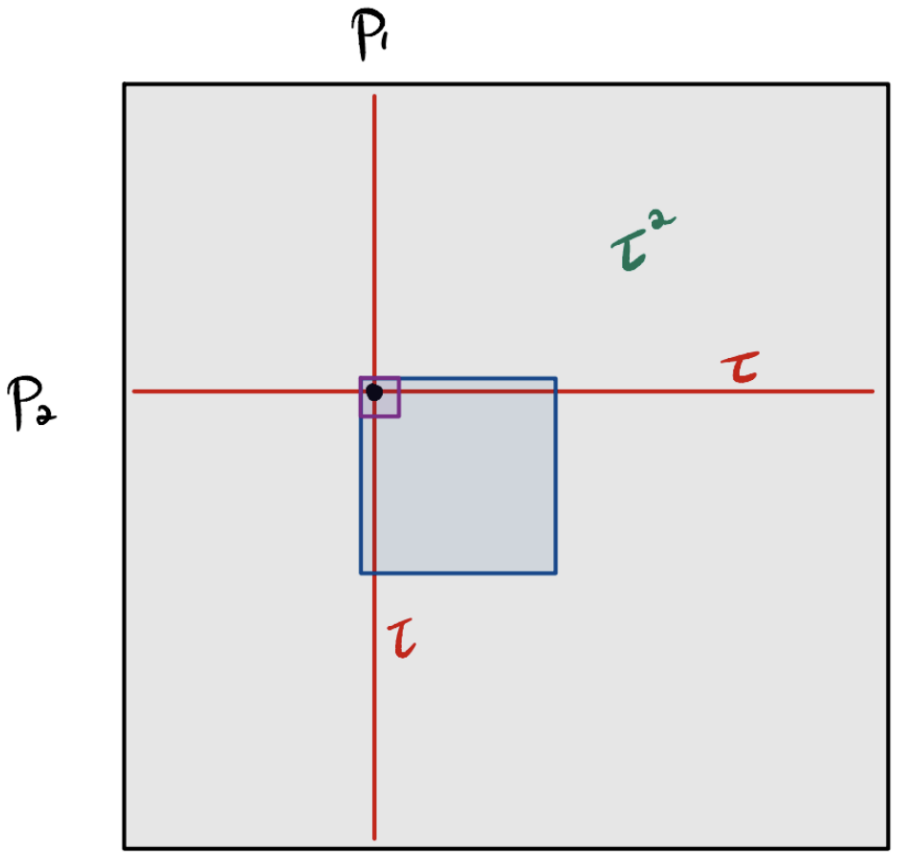
Claim

$(L', \mathcal{D}')$ *is a* (q^2, τ, ρ')*-Dual LCC with* $\rho' = 2\rho - \rho^2$.

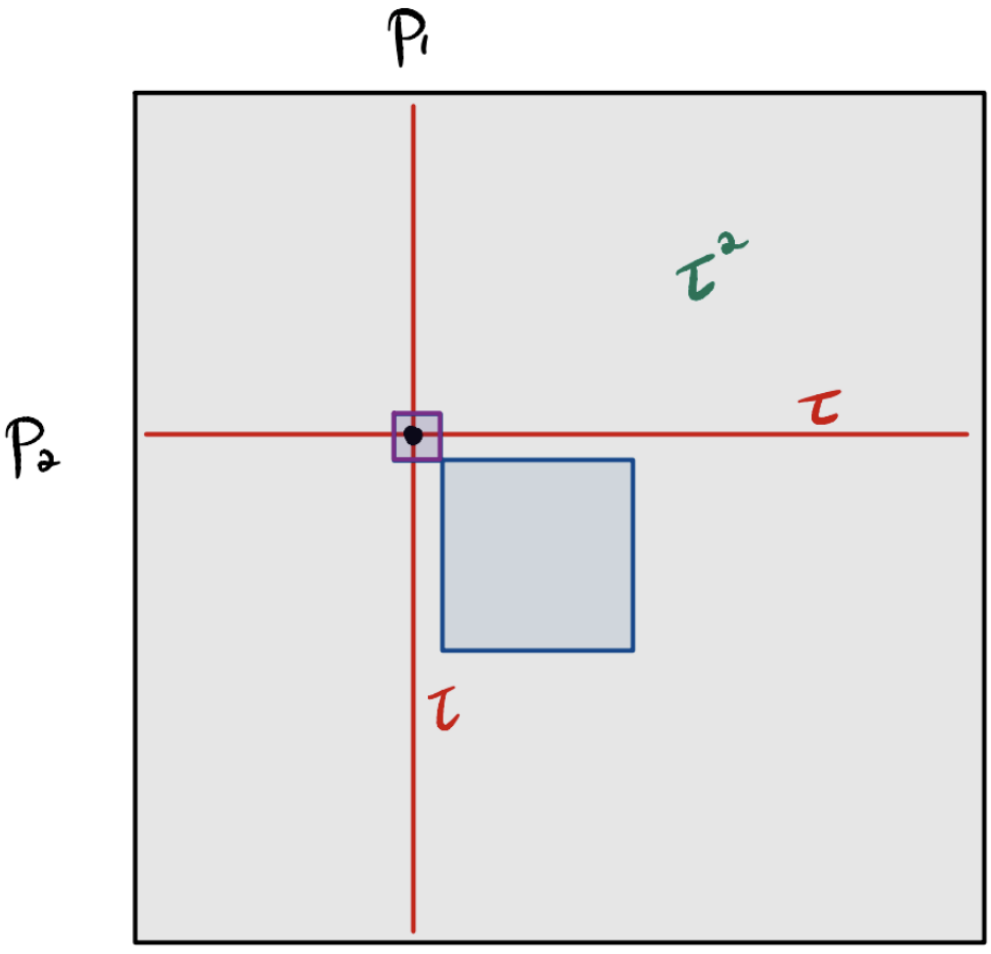
Rate amplification for LCC



Rate amplification for LCC



Rate amplification for LCC



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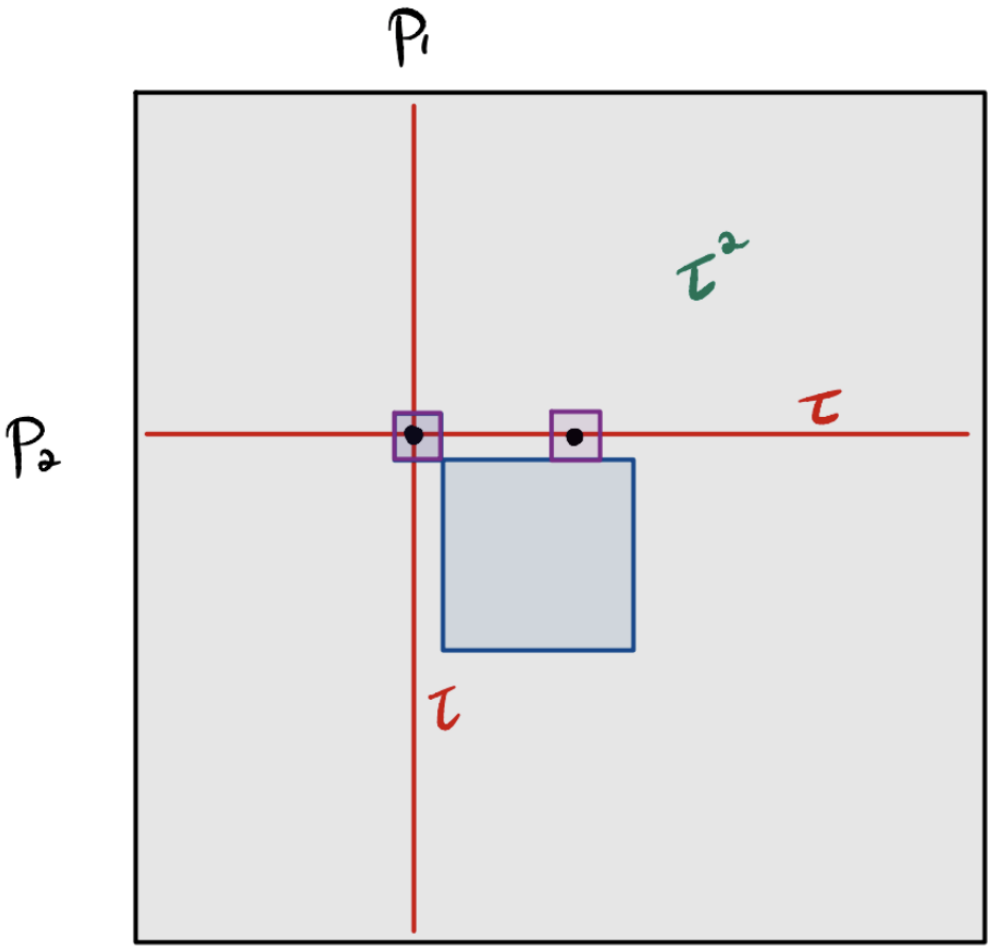
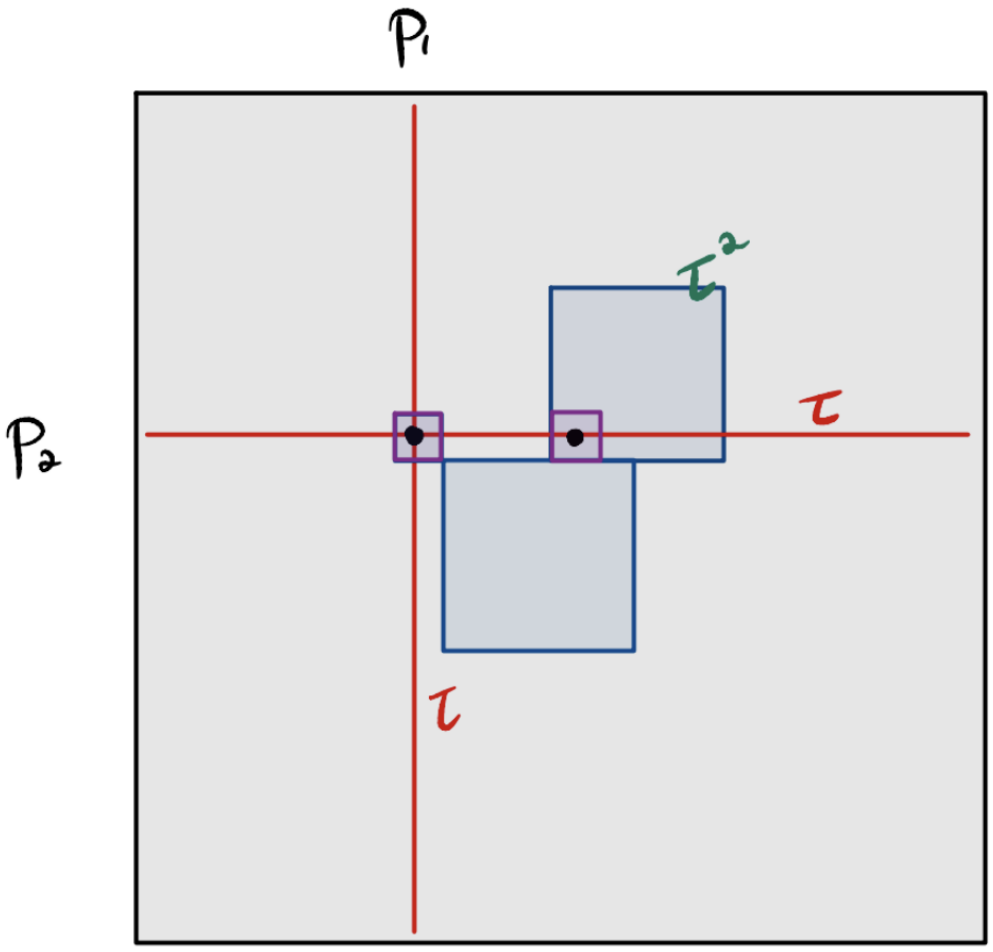
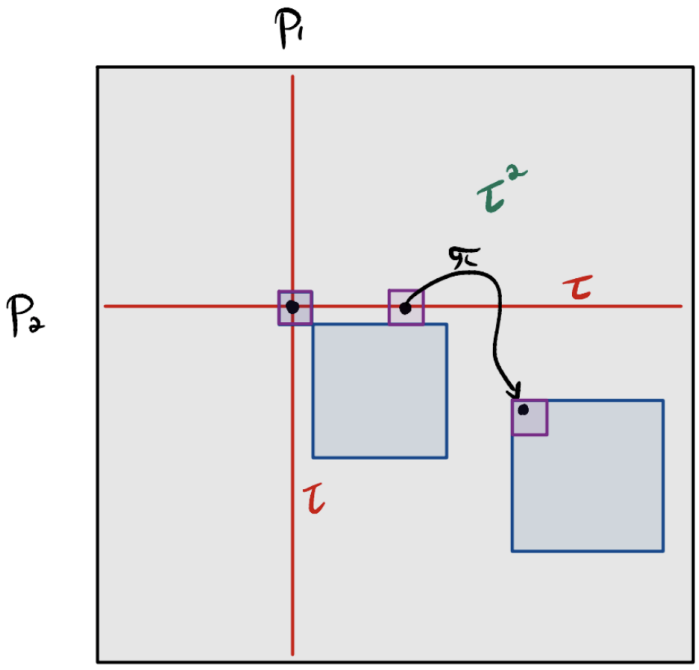


Rate amplification for LCC



Rate amplification for LCC

Let $\pi : P^2 \to P^2$ be a matching. Adjoin $\{e_p + e_{\pi(p)}\}_{p \in P^2}$ to L.



π should be "axis evasive": If $\mathcal{X}_p$ is the set of points of distance 1 from p then

$$\forall p \neq p' \in P^2 \quad |\pi(\mathcal{X}_p) \cap \mathcal{X}_{p'}| \leq s.$$

Rate amplification for LCC

More generally, one can consider a partition π of P^2 s.t.

1. Each part has size c.
2. $\forall p \neq p' \in P^2 \quad |\pi(\mathcal{X}_p) \cap \mathcal{X}_{p'}| \leq s$.

Using such a partition

$$\rho_{\text{new}} = 2\rho - \rho^2 - \frac{1}{c}$$
$$q_{\text{new}} = O(cq^3)$$
$$\tau_{\text{new}} = O(csq \cdot \tau^2).$$

Taking $c = o(\frac{1}{\rho})$ and applying this for $\approx \log \frac{1}{\rho}$ times yields a constant rate.

Axis-evasive partitions

- A probabilistic argument only gives axis-evasive partitions for $c = \Omega(\log |P|)$, with $s = c$.
- We give an explicit construction for certain values of $c, |P|$ with $s = O(c^2)$.
- More precisely, we require $|P| = q$ to be a prime power and $c + 1 \mid q^2 - 1$.

Axis-evasive partitions

- Identify $\mathbb{F}_{q^2}$ with $\mathbb{F}_q^2$.
- Fix $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.
- Let $\beta \in \mathbb{F}_{q^2}$ be an element of order $c+1$ in $\mathbb{F}_{q^2}^\times$.
- The partition is given by the quotient group $\mathbb{F}_{q^2}^\times / \langle \beta \rangle$.
- There are some technical conditions on α, β that can always be met.

Summary

This work was about

$$\text{Bad LC} \implies \text{good LC}$$

Future research

1. Construct a bad (but not so bad) LC.
2. Can it be that the current $2^{\widetilde{O}(\sqrt{\log n})}$ bound is tight?
3. Devise an improved distance amplification procedure for LCC.
4. Improve our rate amplification procedure.
5. Amplify rate under more general conditions.

Thank you!