Spectral-Graph-Theory
Recitation 01.

Recall: $L_G = D_G - M_G$

$x L_G x^t = \sum_{(u,v)\in E} (x(u) - x(v))^2$

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Ex 01: Compute the eignvalues and eignvectors of L_{K_n}

Sol: Claim: $\forall G$ $\vec{1} = (1, \ldots, 1)$ is an eignvector of L_G, with eigenvalue 0.

$$L_G \vec{1} = D_G \vec{1} - M_G \vec{1} = \begin{pmatrix} d(v_1) \\ \vdots \\ d(v_n) \end{pmatrix} - \begin{pmatrix} d(v_1) \\ \vdots \\ d(v_n) \end{pmatrix} = \vec{0} \quad \boxtimes$$

e.val $= 0$ e.vec $= \vec{1}$

Now every other eign. vec, ψ, $\langle \vec{1}, \psi \rangle = 0$

$\Leftrightarrow (*) \sum_{i=1}^{n} \psi(i) = 0$. Let $\psi \in \mathbb{1}^{\perp}$

Formula:

$$\boxed{(L_G \psi)(i) = \sum_{(i,v)\in E} (\psi(i) - \psi(v))}$$

$$\underset{G=K_n}{=} (n-1)\psi(i) - \underbrace{\sum_{v \neq i} \psi(v)} = (n-1)\psi(i) + \psi(i) = n\psi(i)$$

$L_G \psi = n \cdot \psi$. $\forall \psi \in \mathbb{1}^{\perp}$, ψ eignvec, eign. val n.

Val $= \{0, n\}$

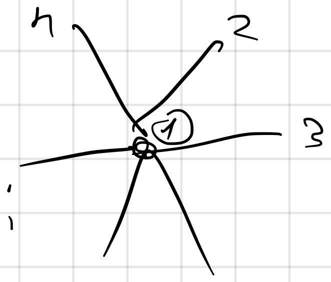
eignVec $= \{\mathbb{1}, \mathbb{1}^{\perp}\}$.

Ex 02: Let $G=([n],E)$, $E=\{(1,i) \mid 1<i\le n\}$

(Star-graph)

Compute the eignvectors and eignvalues of L_G.

Sol: eig. val $=0$, eign vec $\mathbb{1}^t$



$$L_G = D_G - M_G \Rightarrow D_G = \begin{pmatrix} n-1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}, \quad M_G = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & & \\ \vdots & & \ddots & \\ 1 & & & 0 \end{pmatrix}$$

$$L_G = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{pmatrix}, \quad \underline{e_i - e_j}, \quad i \neq j > 1$$

$$D_G(e_i - e_j) = e_i - e_j, \quad M_G(e_i - e_j) = 0$$

$\Rightarrow L_G(e_i - e_j) = e_i - e_j$. eigen vec, eign val $\mathbb{1}$.

We found $n-2$ eigenvec (lin. indp).

$$\left.\begin{array}{l} Tr(L_G) = \sum_{i=1}^{n} \lambda_i = (n-1) + (n-1)\cdot 1 = 2n-2 \\ \quad \rightarrow 0 + (n-2)\,1 + \lambda_n \end{array}\right\} \lambda_n = n.$$

$\varphi = (x, 1, \dots, 1) \Rightarrow x = -(n-1)$

$\varphi_n = (-(n-1), 1, \dots, 1).$

Def: Let $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$.

we define $H_1 \times H_2 := (V_1 \times V_2, E_\times)$

$$E_\times = \{ ((a,b),(c,d)) \mid a = c \wedge (b,d) \in E_2 \text{ or } b = d \wedge (a,c) \in E_1 \}$$

Thm: Let H_1, H_2 (n, m). Assume L_{H_1} has eign. val $\lambda_1, \dots, \lambda_n$ and eignvec $v_1, \dots, v_n$. L_{H_2} " $\mu_1, \dots, \mu_m$

" " $u_1, \dots, u_m$

then $L_{H_1 \times H_2}$ has eignval $\{\mu_i + \lambda_j\}_{i \in [m], j \in [m]}$

" eignvec $(\ \ ,\ )$ w_{ij} $j \in [n]$

$w_{i,j}(k, \ell, \dots$

Proof: $\left(L_{H_1 \times H_2} w_{ij}\right)(k,\ell) = \sum_{(k,\ell)(s,t) \in E_\times} \left(w_{ij}(k,\ell) - \dots (s,t) \right)$

$$= \underbrace{\sum_{(\ell,t) \in E_2} w_{ij}(k,\ell) - w_{ij}(k,t)}_{v_i(k) \cdot u_j(\ell) - v_i(k) \cdot u_j(t)} + \sum_{(k,s) \in E_1} w_{ij}(k,\ell) - w_{ij}(s,\ell)$$

$$= v_i(k) \underbrace{\left(\sum_{(\ell,t) \in E_2} u_j(\ell) - u_j(t) \right)}_{} + \underbrace{u_j(\ell) \left(\sum v_i(k) - v_i(s) \right)}_{}$$

$$v_i(k) \cdot \mu_j u_j(\ell) + u_j(\ell) \cdot \lambda_i v_i(k)$$

$$(\mu_j + \lambda_i)(v_i(k) u_j(\ell)) = (\mu_j + \lambda_i)(w_{ij}(k,\ell)).$$