Spectral Graph Theory		**Winter 2020**
	Problem Set 2	
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Problem 1. Let G be a connected graph. Prove the following claims (the items are independent):

(a) $|\text{Spec}(M_G)| \geq \text{diam}(G) + 1$, where $\text{diam}(G)$ is the diameter of G.

(b) G is bipartite $\iff$ $\text{Spec}(M_G)$ is symmetric about 0.

Problem 2. Let $G = (V, E)$ be graph, and let $G' = (V, E \cup \{e\})$ where $e \notin E$ is a new edge. Find the least constant c (independent of G, G') such that $\lambda_2(G') \leq \lambda_2(G) + c$. You do not need to prove the tightness of your constant c.

Problem 3. Let P be the characteristic polynomial of the adjacency matrix of a graph G. For $v \in V$ denote by P_v the characteristic polynomial of the adjacency matrix of the graph obtained by removing v from G. Prove that

$$P'(x) = \sum_{v \in V} P_v(x).$$

Problem 4. Let $G(\Gamma, S)$ be a Cayley graph over the group Γ and a set of generators S. Let $\chi : G \to \mathbb{C}^*$ be a character, and define the vector $v_\chi(g) = \chi(g)$.

(a) Prove that v_χ is an eigenvector of M_G what is the corresponding eigenvalue?

(b) Conclude that v_χ is an eigenvector of L_G what is the corresponding eigenvalue?

(c) Prove that $\langle v_{\chi_1}, v_{\chi_2} \rangle = 0 \iff \chi_1 \neq \chi_2$.

(d) Describe an orthogonal basis of real vectors (i.e. in $\mathbb{R}^n$) of eigenvectors of L_G.

(e) Present $K_{n,n}$ as a Cayley graph and compute $\text{Spec}(L_{K_{n,n}})$.

Problem 5. In this question you will construct a baby version of what we will later call an *expander graph*. We say that a graph G is a λ*-spectral expander* if λ_2–the second smallest eigenvalue of the corresponding laplacian L_G–is at least λ. The typical goal in constructing expanders is the following: Given a degree bound d (e.g., $d = 3$), construct an infinite family of graphs each with maximal degree at most d, such that all graphs in the family are λ-spectral expanders for some constant $\lambda > 0$ (which may depend on d).

In this question you will construct a family of expanders, not quite with a constant degree, but nevertheless with a relatively slowly increasing degree as a function of the number of vertices. Your construction is going to be based on certain Cayley graphs.

To get you started, let p be a prime and $G(\Gamma, S)$ the Cayley graph where $\Gamma = \mathbb{Z}_p^3$ and $S = \{(b, ab, a^2b) \mid a, b \in \mathbb{Z}_p\}$.

1. What is λ_2?
2. Generalize the above construction to obtain, for infinitely many values of n, a graph on n vertices, maximal degree $d = O(\log^5 n)$, and $\lambda_2 \geq d^{3/4}$.