Introduction to Algebraic-Geometric Codes **Spring 2019**

Final Exam

Publish Date: 23 June 19 *Due Date: 1 August 19*

Instructions:

- Every item worths 10 points. Note that this means you can get up to 110 points.
- You may use everything we saw in class, recitation and homework as long as you state the claim formally.
- In the first three weeks you must not consult with others. We believe this will give you the time to make your own significant progress. Afterwards, you may schedule office hours with Shir or Gil in which we can discuss your attempts. You may also consult with other students during this period. However, you are asked to write the names of all students with whom you consulted. We stress that consulting with us or with your fellow students during the permitted dates will not affect your grade.

Problem 1

The items in this problem are unrelated.

(a) Prove that all the circles in $\mathbb{C}$ pass through the projective points $(0 : \pm i : 1) \in \mathbb{CP}^2$.

(b) Let K be an algebraically closed field. Let $g(x,y) = y^2 - f(x) \in K[x,y]$. Find a sufficient and necessary condition on $g(x,y)$ so that C_g is not integrally closed. Under your condition, describe an integral element over C_g in $K(Z_g)$ that is not contained in C_g. Prove your answer.

(c) Let K be an algebraically closed field. Prove or disprove: $K[x,y]/\langle xy-1\rangle$ is the integral closure of $K[x]$ in $K(x)[y]/\langle xy-1\rangle$.

Problem 2

(a) Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

where the two horizontal sequences are exact. Prove or disprove:

i) φ is surjective $\Rightarrow \varphi'$ is surjective.

ii) φ is surjective $\Rightarrow \varphi''$ is surjective.

iii) φ is injecive $\Rightarrow \varphi'$ is injecive.

iv) φ is injecive $\Rightarrow \varphi''$ is injecive.

(b) Let A be a Dedekind domain. Let $P \in \mathrm{Max}(A)$. Prove that for every integer $n \geq 1$ the A-module P^{n-1}/P^n is isomorphic to the A-module A/P.

Problem 3

Let F be a field, and A a Dedekind domain such that $F \subset A$. Denote $K = \mathrm{Frac}(A)$. Let L/K be a finite separable extension, and denote by B the integral closure of A in L.

(a) Let $v \in \mathcal{V}(L/F)$. Prove that there is a unique $v' \in \mathcal{V}(K/F)$ and $c \in \mathbb{N}$ such that $v|_K = cv'$. In this case we say that v *lies over* v'.

(b) With the notations of the previous item, prove that if v' is not rational then v is not rational.

(c) Assume $v' = v_P$ is a P-adic valuation of A. Factor $PB = M_1^{e_1} \cdots M_s^{e_s}$, where $M_1, \dots, M_s \in \mathrm{Max}(B)$ and $e_1, \dots, e_s \geq 1$. Justify why does such a factorization exist, and prove that every valuation $v \in \mathcal{V}(L/F)$ that lies over v' equals v_{M_i} for some $i \in \{1, \dots, s\}$.

(d) From this point let $F = \mathbb{F}_2$. Consider the absolutely irreducible polynomial $f(x, y) = y^2 + y + x^3 \in F[x, y]$. Let $L = F(Z_f)$. Find all rational points $v \in \mathcal{V}(L/F)$ that lie over P-adic valuations of $F[x]$. Prove your answer.

(e) Prove that there is a unique valuation $v_\infty^f \in \mathcal{V}(L/F)$ that lies over $v_\infty \in \mathcal{V}(F(x)/F)$.

(f) One can prove that $v_\infty^f \in \mathcal{V}(L/F)$ is rational. In this item you may use this fact without a proof. Let $(\mathcal{V}(L/F), L/F)$ be the nonsingular complete curve associated with L and let P_∞ be the point associated with v_∞^f. Find a basis for the Riemann-Roch space $\mathcal{L}(6P_\infty)$. Prove your answer.

Good luck!